Problem A.7

Prove the **triangle inequality**: $\|\,(|\alpha\rangle + |\beta\rangle)\,\| \leq \|\alpha\| + \|\beta\|$.

Solution

Use the same strategy as in Problem A.5: Set $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$ and maybe use the fact that $\langle\gamma\,|\,\gamma\rangle \geq 0$ later. Note that for a complex number z, $z + z^* = 2\,\mathrm{Re}\,z$ and $\mathrm{Re}\,z \leq |z|$.

$$\begin{aligned}
\|\,(|\alpha\rangle + |\beta\rangle)\,\|^2 &= \|\gamma\|^2 \\
&= \langle\gamma\,|\,\gamma\rangle \\
&= \langle\gamma|\Big(|\alpha\rangle + |\beta\rangle\Big) \\
&= \langle\gamma\,|\,\alpha\rangle + \langle\gamma\,|\,\beta\rangle \\
&= \langle\alpha\,|\,\gamma\rangle^* + \langle\beta\,|\,\gamma\rangle^* \\
&= \Big[\langle\alpha|\Big(|\alpha\rangle + |\beta\rangle\Big)\Big]^* + \Big[\langle\beta|\Big(|\alpha\rangle + |\beta\rangle\Big)\Big]^* \\
&= \Big(\langle\alpha\,|\,\alpha\rangle + \langle\alpha\,|\,\beta\rangle\Big)^* + \Big(\langle\beta\,|\,\alpha\rangle + \langle\beta\,|\,\beta\rangle\Big)^* \\
&= \langle\alpha\,|\,\alpha\rangle^* + \langle\alpha\,|\,\beta\rangle^* + \langle\beta\,|\,\alpha\rangle^* + \langle\beta\,|\,\beta\rangle^* \\
&= \langle\alpha\,|\,\alpha\rangle + \langle\alpha\,|\,\beta\rangle^* + \langle\alpha\,|\,\beta\rangle + \langle\beta\,|\,\beta\rangle \\
&= \langle\alpha\,|\,\alpha\rangle + 2\,\mathrm{Re}\,\langle\alpha\,|\,\beta\rangle + \langle\beta\,|\,\beta\rangle \\
&\leq \langle\alpha\,|\,\alpha\rangle + 2|\langle\alpha\,|\,\beta\rangle| + \langle\beta\,|\,\beta\rangle \\
&\leq \langle\alpha\,|\,\alpha\rangle + 2\sqrt{\langle\alpha\,|\,\alpha\rangle}\sqrt{\langle\beta\,|\,\beta\rangle} + \langle\beta\,|\,\beta\rangle \qquad \text{(Schwarz Ineq.)} \\
&= \|\alpha\|^2 + 2\|\alpha\|\|\beta\| + \|\beta\|^2 \\
&= (\|\alpha\| + \|\beta\|)^2
\end{aligned}$$

Therefore,

$$\|\,(|\alpha\rangle + |\beta\rangle)\,\|^2 \leq (\|\alpha\| + \|\beta\|)^2,$$

or taking the square root of both sides,

$$\|\,(|\alpha\rangle + |\beta\rangle)\,\| \leq \|\alpha\| + \|\beta\|.$$